κ state solutions of the Dirac equation for the Eckart potential with pseudospin and spin symmetry

2008 J. Phys. A: Math. Theor. 41 065308

(http://iopscience.iop.org/1751-8121/41/6/065308)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **41** (2008) 065308 (8pp) doi:10.1088/1751-8113/41/6/065308

κ state solutions of the Dirac equation for the Eckart potential with pseudospin and spin symmetry

A Soylu[1,2], O Bayrak[2,3] and I Boztosun[2]

[1] Department of Physics, Nigde University, 51350, Nigde, Turkey
[2] Department of Physics, Erciyes University, 38039, Kayseri, Turkey
[3] Department of Physics, Bozok University, 66200, Yozgat, Turkey

Received 30 March 2007, in final form 1 January 2008
Published 29 January 2008
Online at stacks.iop.org/JPhysA/41/065308

Abstract

For any spin–orbit coupling term κ, the analytical solutions of the Dirac equation for the Eckart potential are presented by using the asymptotic iteration method within the framework of the spin and pseudospin symmetry concept. The energy eigenvalues are obtained in the closed form by applying an approximation to the spin–orbit coupling potential.

PACS numbers: 03.65.Fd, 03.65.Ge, 34.20.Cf, 34.20.Gj

1. Introduction

Recently, there has been a renewed interest in solving the Dirac equation analytically to describe the relativistic spin-1/2 particles. It is well known that the analytical solution of the Dirac equation for $\kappa \neq 0$ can be possible for only few potentials such as the Coulomb [1–3] and the harmonic oscillators [3, 4]. For the case $\kappa = 0$, the analytical solutions can be obtained for a number of potentials. Alhaidari [5] has investigated relativistic extensions of shape-invariant potential classes: Rosen–Morse, Eckart, Pöschl Teller, Scarf as well as the Morse [6] and Hulthén [7] potentials for $\kappa = 0$. Recently, Zou [8] has analysed the s-wave Dirac equation for the equal Eckart scalar and vector potentials by using the supersymmetric quantum mechanics approach and the functional analysis method. Using the same methods, Jia [9] has dealt with the s-wave Dirac equation for the Eckart potential with spin and pseudospin symmetry. The spin and pseudospin symmetric solutions of the Dirac equation are very important to describe the nuclear shell structure [10, 11] and these have been observed in several nuclei for a few potentials such as the harmonic oscillator [12–16], the Morse [17, 18] and the Wood–Saxon potentials [19, 20].

In this paper, our aim is to solve the Dirac equation with the Eckart potential for any κ-state by using a different and more practical method, called the asymptotic iteration method (AIM) [21] within an approximation to spin–orbit coupling potential in order to obtain the relativistic bound state eigenvalues and the corresponding Dirac spinors by spin symmetry and pseudospin

symmetry concept. In the next section, we present AIM with all necessary formulae to perform our calculations. In section 3, a brief introduction of the usual Dirac formalism is presented. In section 3.1, we present the spin symmetric solution of the Dirac–Eckart problem by using AIM for any κ-state and in section 3.2, we also investigate the relativistic bound state eigenvalues and the corresponding spinors of Dirac particles for $\kappa \neq 0$ by using AIM with the pseudospin symmetric case. Finally, section 4 is devoted to the summary and conclusion.

2. The asymptotic iteration method

AIM is proposed [21, 22] and applied [23–26] to solve second-order differential equations of the form

$$y_n''(x) = \lambda_0(x) y_n'(x) + s_0(x) y_n(x), \tag{1}$$

where $\lambda_0(x) \neq 0$ and the prime denote the derivative with respect to x, the extra parameter n is thought as a radial quantum number (see section 3). The variables, $s_0(x)$ and $\lambda_0(x)$ are sufficiently differentiable. To find a general solution to this equation, we differentiate equation (1) with respect to x, we find

$$y_n'''(x) = \lambda_1(x) y_n'(x) + s_1(x) y_n(x), \tag{2}$$

where

$$\lambda_1(x) = \lambda_0'(x) + s_0(x) + \lambda_0^2(x), \qquad s_1(x) = s_0'(x) + s_0(x)\lambda_0(x). \tag{3}$$

Equation (1) can be easily iterated up to $(k+1)$th and $(k+2)$th derivatives, $k = 1, 2, 3, \ldots$. Therefore, we have

$$\begin{aligned} y_n^{(k+1)}(x) &= \lambda_{k-1}(x) y_n'(x) + s_{k-1}(x) y_n(x), \\ y_n^{(k+2)}(x) &= \lambda_k(x) y_n'(x) + s_k(x) y_n(x), \end{aligned} \tag{4}$$

where

$$\begin{aligned} \lambda_k(x) &= \lambda_{k-1}'(x) + s_{k-1}(x) + \lambda_0(x)\lambda_{k-1}(x), \\ s_k(x) &= s_{k-1}'(x) + s_0(x)\lambda_{k-1}(x), \end{aligned} \tag{5}$$

which are called the recurrence relations. From the ratio of the $(k+2)$th and $(k+1)$th derivatives, we have

$$\frac{\mathrm{d}}{\mathrm{d}x} \ln\left[y_n^{(k+1)}(x)\right] = \frac{y_n^{(k+2)}(x)}{y_n^{(k+1)}(x)} = \frac{\lambda_k(x)\left[y_n'(x) + \frac{s_k(x)}{\lambda_k(x)} y_n(x)\right]}{\lambda_{k-1}(x)\left[y_n'(x) + \frac{s_{k-1}(x)}{\lambda_{k-1}(x)} y_n(x)\right]}. \tag{6}$$

For sufficiently large k, if

$$\frac{s_k(x)}{\lambda_k(x)} = \frac{s_{k-1}(x)}{\lambda_{k-1}(x)} = \alpha(x), \tag{7}$$

which is the 'asymptotic' aspect of the method, then, equation (6) is reduced to

$$\frac{\mathrm{d}}{\mathrm{d}x} \ln\left[y_n^{(k+1)}(x)\right] = \frac{\lambda_k(x)}{\lambda_{k-1}(x)}, \tag{8}$$

which yields

$$y_n^{(k+1)}(x) = C_1 \exp\left(\int^x \frac{\lambda_k(x_1)}{\lambda_{k-1}(x_1)}\,\mathrm{d}x_1\right) = C_1 \lambda_{k-1}(x) \exp\left(\int^x [\alpha(x_1) + \lambda_0(x_1)]\,\mathrm{d}x_1\right), \tag{9}$$

where C_1 is the integration constant and the right hand side of equation (9) is obtained by using equations (7) and (8). By inserting equation (9) into equation (4), the first-order differential equation is obtained as

$$y_n'(x) + \alpha(x) y_n(x) = C_1 \exp\left(\int^x [\alpha(x_1) + \lambda_0(x_1)]\,\mathrm{d}x_1\right). \tag{10}$$

This first order differential equation can easily be solved and the general solution of equation (1) can be obtained as

$$y_n(x) = \exp\left(-\int^x \alpha(x_1)\,\mathrm{d}x_1\right)\left[C_2 + C_1\int^x \exp\left(\int^{x_1}[\lambda_0(x_2)+2\alpha(x_2)]\,\mathrm{d}x_2\right)\mathrm{d}x_1\right]. \quad (11)$$

For a given potential, the radial Schrödinger equation is converted to the form of equation (1). Then, $s_0(x)$ and $\lambda_0(x)$ are determined and $s_k(x)$ and $\lambda_k(x)$ parameters are calculated by the recurrence relations given by equation (5).

The termination condition of the method in equation (7) can be arranged as

$$\delta_k(x) = \lambda_k(x)s_{k-1}(x) - \lambda_{k-1}(x)s_k(x) = 0 \qquad k = 1, 2, 3, \ldots \quad (12)$$

where k shows the iteration number. For the exactly solvable potential cases, using equation (12) the roots ε of $\delta_k(x, \varepsilon)$ are independent of x, and the vanishing of δ_k gives us the exact analytical eigenvalues as in equation (32) and the radial quantum number n is equal to the iteration number k [21–23]. For nontrivial potential cases that have no exact solutions, $\delta_k(x, \varepsilon)$ depends on both x and ε. Then, equation $\delta_k(x, \varepsilon) = 0$ is solved for a suitable chosen point $x = x_0$, the choice of which affects the convergence rate of the iteration. A suitable choice of x gives the eigenvalue for small iteration numbers. If it is not chosen well, then the energy eigenvalues may be obtained at large iteration numbers k. A suitable point x_0 may be determined generally as the maximum value of the asymptotic wavefunction or the minimum value of the potential [24–26] and the approximate energy eigenvalues are obtained from the roots of equation (12) for sufficiently great values of k with iteration, for which k is always greater than n in these numerical solutions.

The general solution of equation (1) is given by equation (11). The first part of equation (11) gives us the solutions that are convergent and physical, whereas the second part of equation (11) gives us non-physical solutions that are divergent. Although equation (11) is the general solution of equation (1), we take the coefficient of the second part (C_1) as zero, in order to find the square integrable solutions. Therefore, the corresponding eigenfunctions can be derived from the following wavefunction generator for exactly solvable potentials:

$$y_n(x) = C_2 \exp\left(-\int^x \frac{s_n(x_1)}{\lambda_n(x_1)}\,\mathrm{d}x_1\right), \quad (13)$$

where n represents the radial quantum number.

3. Analytical solution of the Eckart potential

The Dirac wave equation [1, 16, 19, 20, 27] for a single particle with mass M in a scalar potential $S(\vec{r})$ and a vector potential $V(\vec{r})$ can be given as (in units $\hbar = c = 1$)

$$[\vec{\alpha}\cdot\vec{p} + \beta(M + S(\vec{r})) + V(\vec{r})]\psi(\vec{r}) = E\psi(\vec{r}), \quad (14)$$

where $\vec{p}$ and E are the momentum operator and the total relativistic energy of the system, respectively. The total angular momentum operator $\hat{J}$ and spin–orbit matrix operator $\hat{K} = -\beta(\hat{\sigma}\cdot\hat{L} + 1)$ commute with the Dirac Hamiltonian for only spherically symmetric potentials. $\vec{\alpha}$ and β are 4×4 Dirac matrices i.e,

$$\vec{p} = -i\vec{\nabla}, \qquad \vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma}_i \\ \vec{\sigma}_i & 0 \end{pmatrix}, \qquad \text{and} \qquad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad (15)$$

where I is the 2×2 unit matrix and $\vec{\sigma}_{i=x,y,z}$ are 2×2 Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (16)$$

The Dirac spinors may be written according to upper (large) $f_{n\kappa}$ and lower (small) $g_{n\kappa}$ as,

$$\psi_{n\kappa}(\vec{r}) = \begin{pmatrix} f_{n\kappa} \\ g_{n\kappa} \end{pmatrix} = \begin{pmatrix} \frac{F_{n\kappa}(r)}{r} Y^{\ell}_{jm}(\theta, \phi) \\ \frac{iG_{n\kappa}(r)}{r} Y^{\tilde{\ell}}_{jm}(\theta, \phi) \end{pmatrix}, \tag{17}$$

where $Y^{\ell}_{jm}(\theta, \phi)$ and $Y^{\tilde{\ell}}_{jm}(\theta, \phi)$ are the spin and pseudospin spherical harmonics. n is the radial quantum number and m is the projection of the angular momentum on the z axis. The orbital angular momentum quantum numbers l and $\tilde{l}$ refer to the spin and pseudospin quantum numbers, respectively. For a given spin–orbit coupling term $\kappa = \pm 1, \pm 2, \ldots$, the total angular momentum, the orbital angular momentum and pseudoorbital angular momentum are given by $j = |\kappa| - 1/2$, $l = |\kappa + 1/2| - 1/2$ and $\tilde{l} = |\kappa - 1/2| - 1/2$, respectively. Substituting equation (17) to equation (14), we can immediately obtain two coupled ordinary differential equations for the radial parts of the Dirac eigenfunctions, namely

$$\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\kappa}{r}\right) F_{n\kappa}(r) = [M + E_{n\kappa} - V(r) + S(r)]\, G_{n\kappa}(r), \tag{18}$$

and

$$\left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{\kappa}{r}\right) G_{n\kappa}(r) = [M - E_{n\kappa} + V(r) + S(r)]\, F_{n\kappa}(r). \tag{19}$$

By eliminating $G_{n\kappa}(r)$ in equation (18) and $F_{n\kappa}(r)$ in equation (19), we immediately obtain the second-order differential equation for the lower and upper components of the Dirac wavefunction. These are

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{\kappa(\kappa - 1)}{r^2} - (M + E_{n\kappa} - \triangle(r))(M - E_{n\kappa} + \Sigma(r)) - \frac{\frac{\mathrm{d}\Sigma}{\mathrm{d}r}\left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{\kappa}{r}\right)}{M - E_{n\kappa} + \Sigma(r)}\right] G_{n\kappa}(r) = 0 \tag{20}$$

and

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{\kappa(\kappa + 1)}{r^2} - (M + E_{n\kappa} - \triangle(r))(M - E_{n\kappa} + \Sigma(r)) + \frac{\frac{\mathrm{d}\Delta}{\mathrm{d}r}\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\kappa}{r}\right)}{M + E_{n\kappa} - \triangle(r)}\right] F_{n\kappa}(r) = 0, \tag{21}$$

where $\Sigma(r) = V(r) + S(r)$ and $\triangle(r) = V(r) - S(r)$.

3.1. Spin symmetric solution of the Eckart potential for any κ state

In the case of exact spin symmetry $\left(\frac{\mathrm{d}\Delta(r)}{\mathrm{d}r} = 0, \text{i.e. } \Delta(r) = C = \text{const}\right)$, equation (21) becomes

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{\kappa(\kappa + 1)}{r^2} - (M + E_{n\kappa} - C)(M - E_{n\kappa} + \Sigma(r))\right] F_{n\kappa}(r) = 0, \tag{22}$$

where $\kappa = \ell$ for $\kappa < 0$ and $\kappa = -(\ell + 1)$ for $\kappa > 0$. The energy eigenvalues depend on n and ℓ, i.e., $E_{n\kappa} = E(n, \ell(\ell + 1))$, which is well known as the exact spin symmetry [19, 28]. We assume that $\Sigma(r)$ is the Eckart potential [29], which is defined as

$$V(r) = V_1 \operatorname{cosech}^2(\alpha r) - V_2 \coth(\alpha r), \tag{23}$$

where α is the screening parameter, determining the range for the Eckart potential. The Eckart potential can be written in the exponential form:

$$V(r) = 4V_1 \frac{\mathrm{e}^{-2\alpha r}}{(1 - \mathrm{e}^{-2\alpha r})^2} - V_2 \frac{1 + \mathrm{e}^{-2\alpha r}}{1 - \mathrm{e}^{-2\alpha r}}. \tag{24}$$

On the other hand, instead of the spin–orbit coupling potential $V_\kappa(r) = \frac{\kappa(\kappa+1)}{r^2}$, we take an approximate spin–orbit coupling one as

$$\widetilde{V}_\kappa(r) = 4\alpha^2\kappa(\kappa+1)\frac{\mathrm{e}^{-2\alpha r}}{(1-\mathrm{e}^{-2\alpha r})^2} \tag{25}$$

similar to [30–35] in the limit of small α and κ.

By inserting the Eckart potential and the approximate spin–orbit coupling one into equation (22) and using the following *ansätze* in order to make the differential equation more compact

$$\begin{gathered} s = \mathrm{e}^{-2\alpha r}, \qquad \varepsilon = \frac{M^2 - E_{n\kappa}^2 - C(M - E_{n\kappa})}{4\alpha^2}, \qquad \beta = -\frac{V_2(M + E_{n\kappa} - C)}{4\alpha^2}, \\ \gamma(\gamma-1) = \kappa(\kappa+1) + \frac{V_1}{\alpha^2}(M + E_{n\kappa} - C). \end{gathered} \tag{26}$$

The Dirac equation can then be reduced to

$$\frac{\mathrm{d}^2\tilde{F}_{n\kappa}(s)}{\mathrm{d}s^2} + \frac{1}{s}\frac{\mathrm{d}\tilde{F}_{n\kappa}(s)}{\mathrm{d}s} + \left[-\frac{\varepsilon}{s^2} - \frac{\gamma(\gamma-1)}{s(1-s)^2} - \frac{\beta(1+s)}{s^2(1-s)}\right]\tilde{F}_{n\kappa}(s) = 0 \tag{27}$$

The wavefunction should respect the boundary conditions, i.e. $\tilde{F}_{n\kappa}(0) = 0$ at $s = 0$ for $r \longrightarrow \infty$ and $\tilde{F}_{n\kappa}(1) = 0$ at $s = 1$ for $r \longrightarrow 0$. Therefore the reasonable physical wave function we propose is

$$\tilde{F}_{n\kappa}(s) = s^{\sqrt{\varepsilon+\beta}}(1-s)^\gamma f_{n\kappa}(s). \tag{28}$$

If we insert this wavefunction into equation (27), we have the second-order homogeneous linear differential equation in the form

$$\begin{aligned} \frac{\mathrm{d}^2 f_{n\kappa}(s)}{\mathrm{d}s^2} = &\left[\frac{(1+2\gamma+2\sqrt{\varepsilon+\beta})s - (2\sqrt{\varepsilon+\beta}+1)}{s(1-s)}\right] \\ &\times \frac{\mathrm{d}f_{n\kappa}(s)}{\mathrm{d}s} + \left[\frac{2\sqrt{\varepsilon+\beta}\gamma + \gamma^2 + 2\beta}{s(1-s)}\right] f_{n\kappa}(s), \end{aligned} \tag{29}$$

which is now amenable to an AIM solution. By comparing this equation with equation (1), we can write the $\lambda_0(s)$ and $s_0(s)$ values and by means of equation (5), we may calculate $\lambda_k(s)$ and $s_k(s)$. This gives

$$\begin{aligned} \lambda_0(s) &= \frac{(1+2\gamma+2\sqrt{\varepsilon+\beta})s - (2\sqrt{\varepsilon+\beta}+1)}{s(1-s)} \\ s_0(s) &= \frac{2\sqrt{\varepsilon+\beta}\gamma + \gamma^2 + 2\beta}{s(1-s)} \\ \lambda_1(s) &= \frac{(6\sqrt{\varepsilon+\beta}+2+6\sqrt{\varepsilon+\beta}\gamma+3\gamma^2+6\gamma+4\varepsilon+2\beta)s^2 + 2 + 6\sqrt{\varepsilon+\beta}+4\varepsilon+4\beta}{s^2(-1+s)^2} \\ &\quad + \frac{(-12\sqrt{\varepsilon+\beta} - 4\gamma + \gamma^2 - 6\sqrt{\varepsilon+\beta}\gamma - 6\beta + -8\varepsilon - 4)s}{s^2(-1+s)^2} \\ s_1(s) &= \frac{(\gamma^2+2\beta+2\sqrt{\varepsilon+\beta}\gamma)(-2+3s+2\gamma s+2\sqrt{\varepsilon+\beta}s - 2\sqrt{\varepsilon+\beta})}{s^2(-1+s)^2} \\ &\ldots \text{etc.} \end{aligned} \tag{30}$$

By inserting above equations into $\delta_1 = s_0\lambda_1 - s_1\lambda_0 = 0$, we obtain the first δ_1 value as

$$\delta_1 = \frac{(2\beta + 2\sqrt{\varepsilon+\beta} + 2\sqrt{\varepsilon+\beta}\gamma + 1 + 2\gamma + \gamma^2)(2\sqrt{\varepsilon+\beta}\gamma + 2\beta + \gamma^2)}{s^2(-1+s)^2}. \tag{31}$$

The root of equation (31) gives us the first ε_0 value as $\varepsilon_0 = \frac{4\beta^2+\gamma^4}{4\gamma^2}$. Similarly, using the quantization condition given by equation (12) yields other δ and ε values.

$$\begin{aligned}
\delta_2 = s_1\lambda_2 - s_2\lambda_1 = 0 \quad &\Leftarrow \quad \varepsilon_1 = \frac{4\beta^2+\gamma^4+6\gamma^2+4\gamma^3+4\gamma+1}{4(1+\gamma)^2}, \\
\delta_3 = s_2\lambda_3 - s_3\lambda_2 = 0 \quad &\Leftarrow \quad \varepsilon_2 = \frac{4\beta^2+16+32\gamma+24\gamma^2+8\gamma^3+\gamma^4}{4(2+\gamma)^2}, \\
&\ldots \text{etc.}
\end{aligned} \tag{32}$$

When the above expressions are generalized by induction, the eigenvalues turn out to be

$$\varepsilon_{n\kappa} = \frac{4\beta^2+(n+\gamma)^4}{4(n+\gamma)^2} \qquad n = 0, 1, 2, 3, \ldots \tag{33}$$

We should also point out that inserting ε_0 into equation (31), the term $\sqrt{\varepsilon+\beta}$ is equal to $\sqrt{\frac{(2\beta+\gamma^2)^2}{4\gamma^2}}$. Therefore, this equation has two roots, i.e., $\pm\frac{(2\beta+\gamma^2)}{2\gamma}$. The negative root, $-\frac{(2\beta+\gamma^2)}{2\gamma}$, satisfies equation (31). The other root does not satisfy equation (31), which is also valid for $\varepsilon_1, \varepsilon_2, \ldots$, etc.

Using equation (26) and (33), we obtain the energy eigenvalues $E_{n\kappa}$ as

$$(M - E_{n\kappa})(M + E_{n\kappa} - C) = \alpha^2\left[(n+\gamma)^2 + \frac{4\beta^2}{(n+\gamma)^2}\right], \tag{34}$$

where $\gamma = \frac{1}{2} \pm \frac{1}{2}\sqrt{1+4\kappa(\kappa+1)+4\frac{V_1}{\alpha^2}(M+E_{n\kappa}-C)}$. This result is same as the s-state solution of the Dirac–Eckart problem with spin symmetry in [9]. We should clarify that the energy eigenvalues of the Eckart potential for any κ-states are exactly solvable, i.e the roots of equation (12) are independent of the x_0 point.

The corresponding eigenfunctions by using the wavefunction generator given by equation (13) could be obtained as

$$f_{n\kappa}(s) = (-1)^n C_2 \frac{\Gamma(n+2\sqrt{\varepsilon_{n\kappa}+\beta}+1)}{\Gamma(2\sqrt{\varepsilon_{n\kappa}+\beta}+1)} {}_2F_1(-n, 2(\sqrt{\varepsilon_{n\kappa}+\beta}+\gamma)+n, 1+2\sqrt{\varepsilon_{n\kappa}+\beta}; s), \tag{35}$$

where Γ and ${}_2F_1$ are the gamma function and the Gauss hypergeometric function, respectively [36]. Therefore, we can write the total radial wavefunction by using equations (28) and (35) as

$$\tilde{F}_{n\kappa}(s) = N s^{\sqrt{\varepsilon_{n\kappa}+\beta}}(1-s)^{\gamma} {}_2F_1(-n, 2(\sqrt{\varepsilon_{n\kappa}+\beta}+\gamma)+n, 1+2\sqrt{\varepsilon_{n\kappa}+\beta}; s), \tag{36}$$

where N is normalization constant.

3.2. Pseudospin symmetric solution of the Eckart potential for any κ state

In the case of exact pseudospin symmetry ($\frac{d\Sigma(r)}{dr} = 0$, i.e., $\Sigma(r) = C = \text{const}$), equation (20) becomes

$$\left[\frac{d^2}{dr^2} - \frac{\kappa(\kappa-1)}{r^2} - (M + E_{n\kappa} - \triangle(r))(M - E_{n\kappa} + C)\right] G_{n\kappa}(r) = 0, \tag{37}$$

where $\kappa = -\tilde{\ell}$ for $\kappa < 0$ and $\kappa = \tilde{\ell}+1$ for $\kappa > 0$ and $\triangle(r)$ is the Eckart potential. The energy eigenvalues depend on n and $\tilde{\ell}$, i.e., $E_{n\kappa} = E(n, \tilde{\ell}(\tilde{\ell}+1))$. The eigenstates with $j = \tilde{\ell} \pm 1/2$ are degenerate for $\tilde{\ell} \neq 0$, which is well known as the exact pseudospin symmetry [28, 19]. The Dirac equation cannot be solved exactly for the Eckart potential for $\kappa \neq 0$ by using the standard methods. Therefore, in order to obtain an exact analytical solution, an approximation

has to be made for the $\kappa(\kappa-1)/r^2$ term, similar to the previous section. According to this approximation, we use $4\alpha^2\frac{e^{-2\alpha r}}{(1-e^{-2\alpha r})^2}$ instead of $\frac{1}{r^2}$. After inserting equations (23) and (25) into equation (37) and by using the *ansätze*

$$s = e^{-2\alpha r}, \qquad \varepsilon = \frac{M^2 - E_{n\kappa}^2 + C(M + E_{n\kappa})}{4\alpha^2}, \qquad \beta = \frac{V_2(M - E_{n\kappa} + C)}{4\alpha^2},$$
$$\gamma(\gamma-1) = \kappa(\kappa-1) - \frac{V_1}{\alpha^2}(M - E_{n\kappa} + C). \tag{38}$$

The Dirac equation can be reduced to the following form

$$\frac{d^2\tilde{G}_{n\kappa}(s)}{ds^2} + \frac{1}{s}\frac{d\tilde{G}_{n\kappa}(s)}{ds} + \left[-\frac{\varepsilon}{s^2} - \frac{\gamma(\gamma-1)}{s(1-s)^2} - \frac{\beta(1+s)}{s^2(1-s)}\right]\tilde{G}_{n\kappa}(s) = 0. \tag{39}$$

The reasonable physical wavefunction we propose is

$$\tilde{G}_{n\kappa}(s) = s^{\sqrt{\varepsilon+\beta}}(1-s)^{\gamma} f_{n\kappa}(s). \tag{40}$$

Similar to the previous section, we can immediately obtain the energy eigenvalues

$$(M + E_{n\kappa})(M - E_{n\kappa} + C) = \alpha^2\left[(n+\gamma)^2 + \frac{4\beta^2}{(n+\gamma)^2}\right]. \tag{41}$$

where $\gamma = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 + 4\kappa(\kappa-1) - 4\frac{V_1}{\alpha^2}(M + E_n - C)}$. This result is same as the s-state solution of the Dirac–Eckart problem with pseudospin symmetry in [9].

Now, as indicated in section 2, we can determine the corresponding wavefunctions by using equation (13).

$$f_{n\kappa}(s) = (-1)^n C_2 \frac{\Gamma(n + 2\sqrt{\varepsilon_{n\kappa} + \beta} + 1)}{\Gamma(2\sqrt{\varepsilon_{n\kappa} + \beta} + 1)} {}_2F_1(-n, 2(\sqrt{\varepsilon_{n\kappa} + \beta} + \gamma) + n, 1 + 2\sqrt{\varepsilon_{n\kappa} + \beta}; s). \tag{42}$$

Thus, we can write the total radial wavefunction as

$$\tilde{G}_{n\kappa}(s) = N s^{\sqrt{\varepsilon_{n\kappa}+\beta}}(1-s)^{\gamma} {}_2F_1(-n, 2(\sqrt{\varepsilon_{n\kappa} + \beta} + \gamma) + n, 1 + 2\sqrt{\varepsilon_{n\kappa} + \beta}; s), \tag{43}$$

where N is the normalization constant.

4. Conclusion

In this study, we have performed the bound state solution of the Dirac equation for the Eckart potential within the framework of the asymptotic iteration method by applying an approximation to the centrifugal-like term. The spin–orbit coupling potential $V_\kappa(r)$ is close to the approximated potential $\widetilde{V}_\kappa(r)$ for the small α and κ values. Using this approximated potential, we have obtained the energy eigenvalues and Dirac spinors in the closed form for the case of the spin symmetry and exact pseudospin symmetry in a systematic way for arbitrary κ state.

Acknowledgments

This paper is an output of the project supported by the Scientific and Technical Research Council of Turkey (TÜBİTAK), under the project number TBAG-106T024. This work is also partially supported by the Turkish State Planning Organization (DPT) under the grant no DPT-2006K-120470. I Boztosun acknowledges the support of Turkish Academy of Sciences (TÜBA-GEBİP). A Soylu acknowledges the financial support of TÜBİTAK-BİDEP.

References

[1] Grenier W 2000 *Relativistic Quantum Mechanics* 3rd edn (Berlin: Springer)
[2] Ciftci H, Hall R L and Saad N 2005 *Phys. Rev.* A **72** 022101
[3] Alhaidari A D, Bahlouli H and Al-Hasan A 2006 *Phys. Lett.* A **349** 87
[4] Villalba V M 1994 *Phys. Rev.* A **49** 586
Rozmej P and Arvieu R 1999 *J. Phys. A: Math. Gen.* **32** 5367
[5] Alhaidari A D 2001 *J. Phys. A: Math. Gen.* **34** 9827
[6] Alhaidari A D 2001 *Phys. Rev. Lett.* **87** 210405
[7] Alhaidari A D 2004 *J. Phys. A: Math. Gen.* **37** 5805
[8] Zou X, Yi L Z and Jia C S 2005 *Phys. Lett.* A **346** 54
[9] Jia C S, Guo P and Peng X L 2006 *J. Phys. A: Math. Gen.* **39** 7737
[10] Arima A, Harvey M and Shimizu K 1969 *Phys. Lett.* **30B** 517
[11] Hecht K T and Adler A 1969 *Nucl. Phys.* A **137** 129
[12] Lisboa R, Malheiro M, de Castro A S, Alberto P and Fiolhais M 2004 *Phys. Rev.* C **69** 024319
[13] Ginocchio J N 2004 *Phys. Rev.* C **69** 034318
[14] Ginocchio J N 2005 *Phys. Rev. Lett.* **95** 252501
[15] de Castro A S, Alberto P, Lisboa R and Malheiro M 2006 *Phys. Rev.* C **73** 054309
[16] Guo J-Y, Fang X-Z and Xu F-X 2005 *Nucl. Phys.* A **757** 411
[17] Berkdemir C 2006 *Nucl. Phys.* A **770** 32
[18] Qiang W-C, Zhou R-S and Gao Y 2007 *J. Phys. A: Math. Theor.* **40** 1677
[19] Guo J-Y and Sheng Z-Q 2005 *Phys. Lett.* A **338** 90
[20] Xu Q and Zhu S-J 2006 *Nucl. Phys.* A **768** 161
[21] Ciftci H, Hall R L and Saad N 2003 *J. Phys. A: Math. Gen.* **36** 11807
[22] Ciftci H, Hall R L and Saad N 2005 *J. Phys. A: Math. Gen.* **38** 1147
[23] Bayrak O and Boztosun I 2006 *J. Phys. A: Math. Gen.* **39** 6955
[24] Boztosun I, Karakoc M, Yasuk F and Durmus A 2006 *J. Math. Phys.* **47** 062301
[25] Fernández F M 2004 *J. Phys. A: Math. Gen.* **37** 6173
[26] Barakat T 2006 *J. Phys. A: Math. Gen.* **39** 823
[27] Ginocchio J N 2005 *Phys. Rep.* **414** 165
[28] Ginocchio J N and Madland D G 1998 *Phys. Rev.* C **57** 1167
[29] Eckart C 1930 *Phys. Rev.* **35** 1303
[30] Bayrak O, Kocak G and Boztosun I 2006 *J. Phys. A: Math. Gen.* **39** 11521
[31] Gonul B, Ozer O, Cancelik Y and Kocak M 2000 *Phys. Lett.* A **275** 238
[32] Greene R L and Aldrich C 1976 *Phys. Rev.* A **14** 2363
[33] Myhrman U 1983 *J. Phys. A: Math. Gen.* **16** 263
[34] Bechlert A and Bühring W 1988 *J. Phys. B: At. Mol. Opt. Phys.* **21** 817
[35] Qian S-W, Huang B-W and Gu Z-Y 2002 *New J. Phys.* **4** 13
[36] Arfken G B and Weber H J 1995 *Mathematical Methods for Physicists* (New York: Academic)